SYDNEY TECHNICAL HIGH SCHOOL

Celebrating 100 years of public education

Mathematics Extension 2

HSC ASSESSMENT TASK

JUNE 2011

General Instructions

- Working time allowed – 70 minutes
- Write using black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown
- Start each question on a new page
- Attempt all questions

NAME :________________________

TEACHER :____________________

QUESTION 1	QUESTION 2	QUESTION 3	TOTAL

Question 1 (17 marks)

a) Find $\int x^3 \ln x \ dx$ 3

b) Show that $\int_5^6 \frac{x+2}{(x-4)(x-1)} \ dx = \ln\frac{16}{5}$ 3

c) $1+i$ and $3-i$ are zeroes of a monic polynomial, $P(x)$, of degree 4

with real coefficients.

Express $P(x)$ as a product of two real quadratic factors. 2

d) i) Solve the equation $16x^4 - 16x^2 + 1 = 0$ 2

using the identity $cos4\theta = 8cos^4\theta - 8cos^2\theta + 1$.

ii) Solve the equation $16x^4 - 16x^2 + 1 = 0$ as a quadratic in x^2, 3

and hence find the exact value of $\cos\frac{\pi}{12}$. Justify your answer.

e) The equation $x^3 + 3px + q = 0$ has a double root at $x = k$.

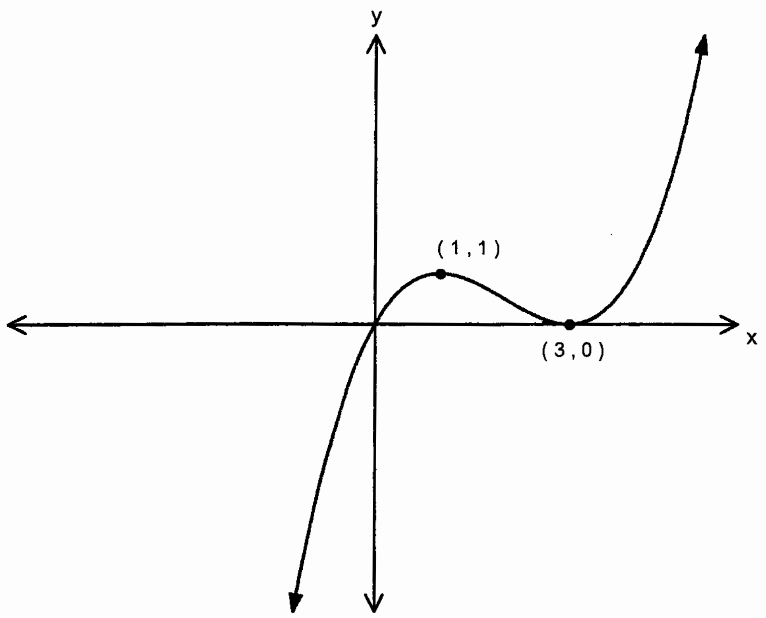
i) Show that $p = -k^2$ 1

ii) Hence solve $x^3 - 6ix + 4 - 4i = 0$ given that it has a double root. 3

Question 2 (17 marks) - Start a new page

a) Below is a sketch of the function $y = g(x)$.

There are stationary points at $(1, 1)$ and $(3, 0)$.



On separate diagrams, draw neat sketches of the following, clearing showing any important features or points.

i) $y = \int g(x)\, dx$ 2

ii) $y = \frac{1}{g(x+1)}$ 2

iii) $y^2 = g(x)$ 2

iv) $y = \cos^{-1} g(x)$ 2

Question 2 (continued)

b) The points $P(2p, \frac{2}{p})$ and $Q(2q, \frac{2}{q})$ lie on the hyperbola $xy = 4$.

i) What is the eccentricity of the hyperbola $xy = 4$. **1**

ii) Find the equation of the chord PQ. **2**

iii) Given that the chord PQ always passes through the point $(4, 2)$ show that $pq = p + q - 2$. **1**

iv) Find the locus of M, the midpoint of PQ. **2**

c) Evaluate $\int_0^{\frac{\pi}{6}} \sin 2x \ \sin 4x \, dx$ **3**

Question 3 (17 marks) - Start a new page

a) Find $\int \frac{2x}{x^2+10x+26}\,dx$ 3

b) Find $\int \frac{1+sinx}{cos^2x}\,dx$ 2

c) The equation $2x^4 - 3x^2 - 2x + k = 0$ has a triple root. 3

Find the value of k.

d) If α, β *and* γ are the roots of the equation $x^3 - 6x^2 - 4x + 2 = 0$.

i) Find the monic polynomial equation with roots $2\alpha, 2\beta, 2\gamma$ 2

ii) Find the constant term in the monic polynomial with roots

$$\frac{1+\alpha}{1-\alpha}, \frac{1+\beta}{1-\beta}, \frac{1+\gamma}{1-\gamma}$$ 3

e) Use integration by parts to evaluate $\int_0^\infty 3y\, e^{-y}(1-e^{-y})^2\,dy$ 4

End of paper. ☺

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

QUESTION 1

a) $\int x^3 \ln x \, dx$ $\qquad u = \ln x \qquad u' = \frac{1}{x}$

$v = \frac{1}{4}x^4 \qquad v' = x^3$

$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 \, dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + c$

b) $\int_5^6 \frac{x+2}{(x-4)(x-1)} \, dx$ $\qquad \frac{x+2}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1}$

$\therefore x+2 = A(x-1) + B(x-4)$

$= \int_5^6 \frac{2}{x-4} - \frac{1}{x-1} \, dx$ $\qquad x=1 \Rightarrow 3 = -3B \quad \therefore B = -1$

$A + B = 1 \quad \therefore A = 2$

$= 2\ln(x-4) - \ln(x-1) \Big]_5^6$

$= (2\ln 2 - \ln 5) - (2\ln 1 - \ln 4)$

$= 2\ln 4 - \ln 5$

$= \ln 16 - \ln 5$

$= \ln \frac{16}{5}$

c) roots are $1+i$, $1-i$, $3-i$, $3+i$

Sum	2	6
product	2	10

$\therefore P(x) = (x^2 - 2x + 2)(x^2 - 6x + 10)$

d) i) $16x^4 - 16x^2 + 1 = 0$

$2(8x^4 - 8x^2 + 1) - 1 = 0$

let $x = \cos\theta$

$2(8\cos^4\theta - 8\cos^2 x + 1) - 1 = 0$

$2\cos 4\theta = 1$

$\cos 4\theta = \frac{1}{2}$

$4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

$\therefore x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{11\pi}{12}$

ii) $16x^4 - 16x^2 + 1 = 0$

let $u = x^2$

$16u^2 - 16u + 1 = 0$

$$u = \frac{16 \pm \sqrt{16^2 - 4.16.1}}{32}$$

$$= \frac{16 \pm \sqrt{192}}{32}$$

$$= \frac{16 \pm 8\sqrt{3}}{32}$$

$$= \frac{2 \pm \sqrt{3}}{4}$$

$$\therefore x^2 = \frac{2 \pm \sqrt{3}}{4}$$

$$x = \frac{\pm\sqrt{2 \pm \sqrt{3}}}{2}$$

$\therefore \cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$ (largest positive)

e) i) $x^3 + 3px + q = 0$

$3x^2 + 3p = 0$

$x^2 = -p$

double root of $x = k$

$\Rightarrow \quad k^2 = -p$

$p = -k^2$

ii) roots k, k, α

$x^3 + 3px + q = x^3 - 6ix + 4 - 4i$

$3p = -6i \qquad q = 4 - 4i$

$p = -2i$

$2k + \alpha = 0$

$k^2 + 2k\alpha = -6i$

$k^2\alpha = -4 + 4i$

but $k^2 = 2i$

$\alpha(2i) = -4 + 4i$

$\alpha = \frac{-4+4i}{2i} \times \frac{i}{i}$

$= \frac{-4i - 4}{-2}$

$= 2i + 2$

$\therefore \quad 2k + 2i + 2 = 0$

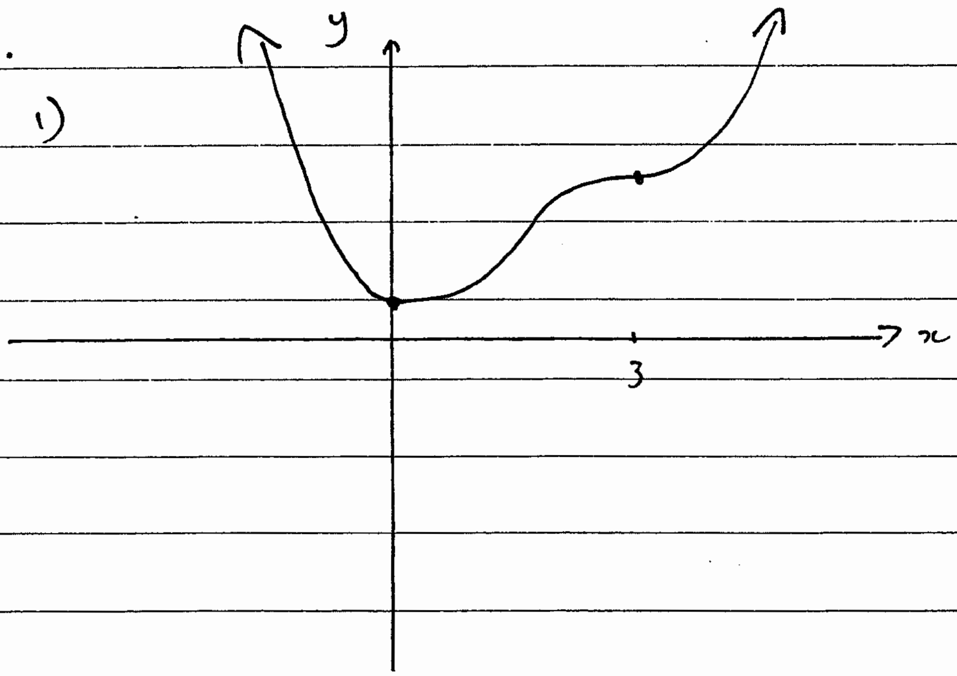
$2k = -2i - 2$

$k = -i - 1$

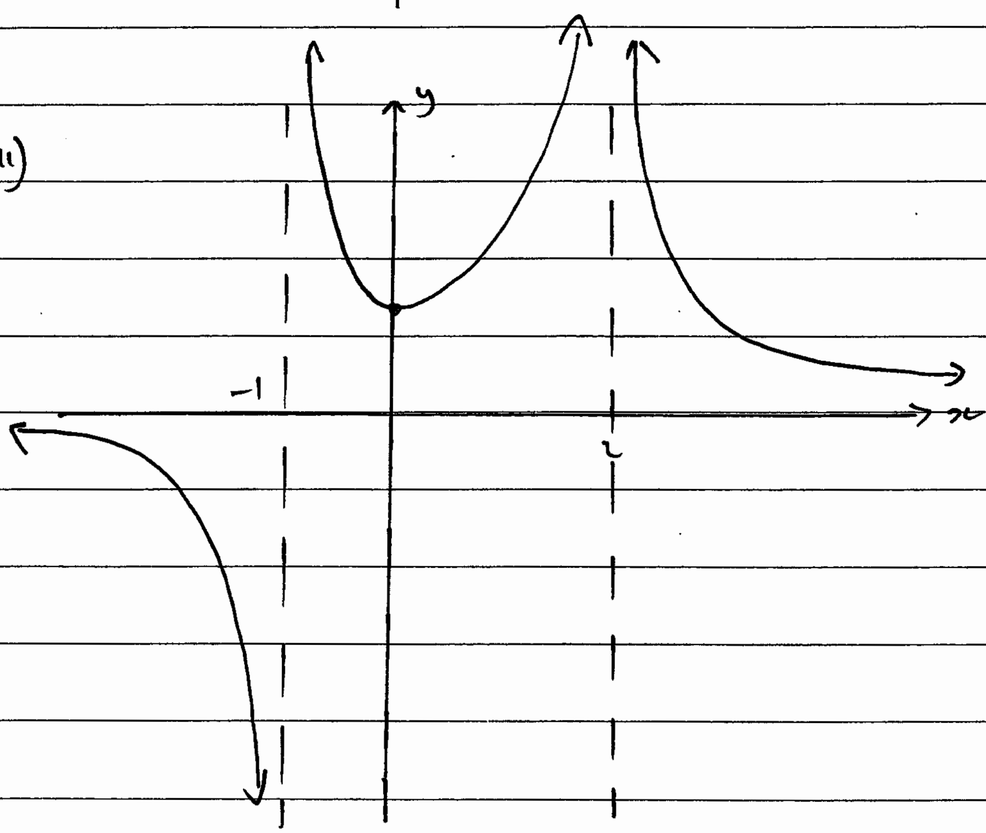
$\therefore \; -i-1, \; -i-1, \; 2i+2$ are solutions

2. a.

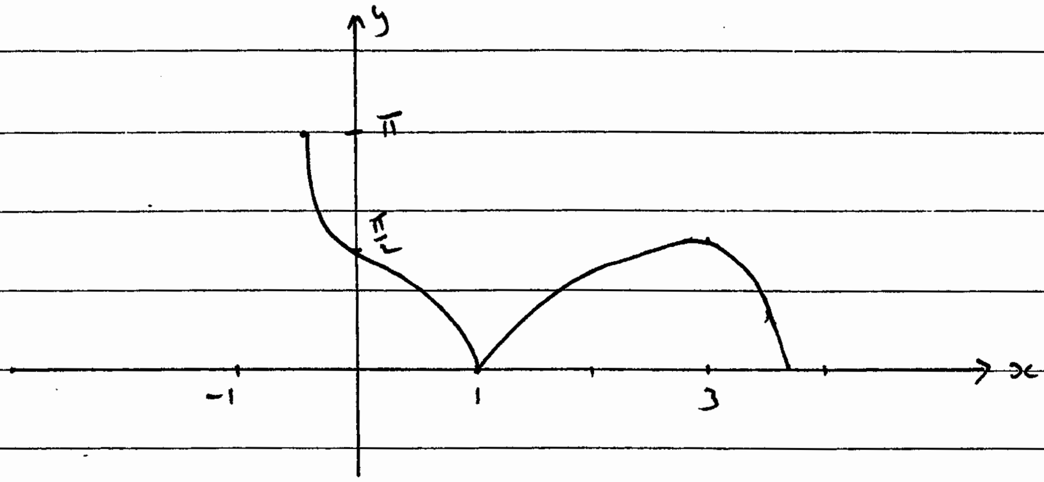
i)



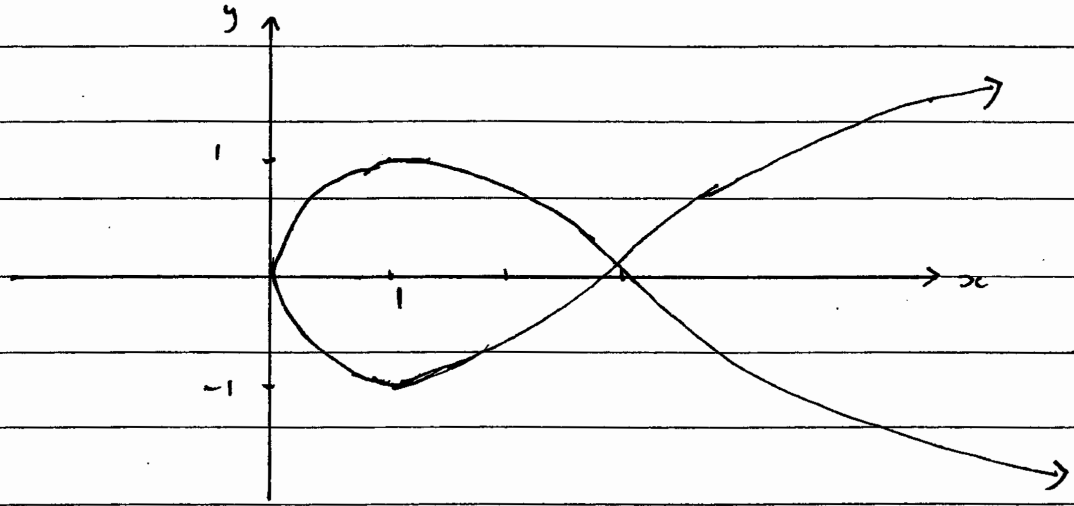
ii)



iii)



iv)



b) i) $e = \sqrt{2}$

ii) $m = \dfrac{\frac{2}{q} - \frac{2}{p}}{2q - 2p}$

$= \dfrac{2p - 2q}{pq(q-p)}$

$= \dfrac{-1}{pq}$

$\therefore \quad y - \dfrac{2}{p} = \dfrac{-1}{pq}(x - 2p)$

$pqy - 2q = -x + 2p$

$x + pqy = 2(p+q)$

iii) sub $x = 4$, $y = 2$

$4 + 2pq = 2(p+q)$

$2 + pq = p+q$

$pq = p+q-2$

iv) midpoint $\left(p+q, \dfrac{p+q}{pq} \right)$

$\therefore \quad x = p+q \quad , \quad y = \dfrac{p+q}{pq}$

$\therefore$ locus $\quad y = \dfrac{p+q}{p+q-2}$

$y = \dfrac{x}{x-2}$

c) $\displaystyle\int_0^{\frac{\pi}{6}} \sin 2x \sin 4x \, dx$ $\qquad \sin 4x = 2\sin 2x \cos 2x$

$= \displaystyle\int_0^{\frac{\pi}{6}} 2 \sin^2 2x \cos 2x \, dx$

$= \left. \frac{1}{3} \sin^3 2x \right]_0^{\frac{\pi}{6}}$

$= \frac{1}{3} \left(\sin^3 \frac{\pi}{3} - \sin^3 0 \right)$

$= \frac{1}{3} \cdot \frac{3\sqrt{3}}{8}$

$= \dfrac{\sqrt{3}}{8}$

QUESTION 3

a) $\int \frac{2x}{x^2+10x+26}\,dx$

$= \int \frac{2x+10}{x^2+10x+26} - \frac{10}{x^2+10x+26}\,dx$

$= \ln(x^2+10x+26) - \int \frac{10}{(x+5)^2+1}\,dx$

$= \ln(x^2+10x+26) - 10\tan^{-1}(x+5) + c$

b) $\int \frac{1+\sin x}{\cos^2 x}\,dx$

$= \int \sec^2 x + \sec x \tan x \, dx$

$= \tan x + \sec x + c$

c) $P(x) = 2x^4 - 3x^2 - 2x + k$

$P'(x) = 8x^3 - 6x - 2$

$P''(x) = 24x^2 - 6$

triple root is root of $P''(x) = 0$

$24x^2 - 6 = 0$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

$P'(\frac{1}{2}) \neq 0 \quad \therefore$ triple root is $-\frac{1}{2}$.

$\therefore \; 2(-\frac{1}{2})^4 - 3(-\frac{1}{2})^2 - 2(-\frac{1}{2}) + k = 0$

$\frac{1}{8} - \frac{3}{4} + 1 + k = 0$

$k = -\frac{3}{8}$

d) i) required polynomial $\Rightarrow P\left(\frac{x}{2}\right)=0$

$$\left(\frac{x}{2}\right)^3-6\left(\frac{x}{2}\right)^2-4\left(\frac{x}{2}\right)+2=0$$

$$\frac{x^3}{8}-\frac{3x^2}{2}-2x+2=0$$

$$x^3-12x^2-16x+16=0$$

ii)

$$y=\frac{1+x}{1-x}$$

$$y-xy=1+x$$

$$y-1=x(y+1)$$

$$x=\frac{y-1}{y+1}$$

$\therefore$ required polynomial $\Rightarrow P\left(\frac{y-1}{y+1}\right)=0$

$$\left(\frac{y-1}{y+1}\right)^3-6\left(\frac{y-1}{y+1}\right)^2-4\left(\frac{y-1}{y+1}\right)+2=0$$

$$(y-1)^3-6(y-1)^2(y+1)-4(y-1)(y+1)^2+2(y+1)^3=0$$

coefficient of: y^3 is $1-6-4+2=-7$

coefficient of constant is $-1-6+4+2=-1$

$\therefore$ constant term $=\frac{1}{7}$

Teacher's Name: ______ **Student's Name/Nº:** ______

e) $\int_0^\infty 3ye^{-y}(1-e^{-y})^2\,dy$

$$\left[\begin{array}{ll} u = 3y & u' = 3 \\ v = \frac{1}{3}(1-e^{-y})^3 & v' = e^{-y}(1-e^{-y})^2 \end{array}\right]$$

$$= \left[y(1-e^{-y})^3\right]_0^\infty - \int_0^\infty (1-e^{-y})^3\,dy$$

$$= \left[y(1-e^{-y})^3\right]_0^\infty - \int_0^\infty 1 - 3e^{-y} + 3e^{-2y} - e^{-3y}\,dy$$

$$= y(1-e^{-y})^3 - y - 3e^{-y} + \frac{3}{2}e^{-2y} - \frac{1}{3}e^{-3y}\Big]_0^\infty$$

$$= (0) - \left(-3 + \frac{3}{2} - \frac{1}{3}\right)$$

$$= \frac{11}{6}$$